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Ground states for a system of Schrödinger equations with critical exponent ☆

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Abstract

We study the following system of nonlinear Schrödinger equations:

$$\begin{cases} -\Delta u + \mu u = |u|^{p-1}u + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{2^*-2}v + \lambda u, & x \in \mathbb{R}^N, \end{cases}$$

where $N \geqslant 3$, $2^* = \frac{2N}{N-2}$, $1 < p < 2^* - 1$ and μ, ν, λ are positive parameters satisfying $0 < \lambda < \sqrt{\mu\nu}$. We show that, there is some critical value $\mu_0 \in (0, 1)$, such that this system has a positive ground state solution if $0 < \mu \leqslant \mu_0$. In the case $\mu > \mu_0$, there exists $\lambda_{\mu,\nu} \in [\sqrt{(\mu - \mu_0)\nu}, \sqrt{\mu\nu})$ such that, this system has no ground state solutions if $\lambda < \lambda_{\mu,\nu}$; while this system has a positive ground state solution if $\lambda > \lambda_{\mu,\nu}$. In particular, if $p = 2^* - 1$, the system has no nontrivial solutions. Some further properties of the ground state solutions are also studied. This seems to be the first result for such a critical Schrödinger system.

Keywords: Ground states; Schrödinger system; Critical exponent

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1. Introduction

In this paper we study the following system of nonlinear Schrödinger equations:

$$\begin{cases} -\Delta u + \mu u = |u|^{p-1}u + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{q-1}v + \lambda u, & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geqslant 3$, μ, ν, λ are positive parameters, $1 < p, q \leqslant 2^* - 1$ and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. Systems of this type arise in nonlinear optics (cf. [1]).

It is well known that a solution $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ of (1.1) is called a bound state. A bound state such that $(u, v) \neq (0, 0)$ is called a nontrivial bound state. A solution is called a ground state if $(u, v) \neq (0, 0)$ and its energy is minimal among the energy of all the nontrivial bound states of (1.1). A ground state satisfying $u > 0$, $v > 0$ is called a positive ground state.

The case of a single equation $-\Delta u + u = |u|^{p-1}u$, $u \in H^1(\mathbb{R}^N)$, has been widely studied by many researchers, and related results can been seen in [5–7,16–18] and the references therein. On the other hand, there are also some results on the linearly coupled system (1.1) in the past several years. In case $N \leqslant 3$, $\mu = \nu = 1$, $p = q = 3$ and $\lambda > 0$ small enough, Ambrosetti, Colorado and Ruiz [3] proved that (1.1) has multi-bump solitons . When $|u|^{p-1}u$ and $|v|^{q-1}v$ are replaced by $f(x, u) = (1 + c(x))|u|^{p-1}u$ and $g(x, v) = (1 + d(x))|v|^{p-1}v$ respectively, system (1.1) has been studied by Ambrosetti [2] with dimension $N = 1$ and Ambrosetti, Cerami and Ruiz [4] with dimension $N \geqslant 2$. When $\mu = \nu = 1$, $1 < p = q < 2^* - 1$ and $0 < \lambda < 1$, Ambrosetti–Cerami–Ruiz proved that system (1.1) has a positive ground state solution (see Section 3 in [4]). Note that this result can be extended to the more general case of $0 < \lambda < \sqrt{\mu\nu}$ and $1 < p, q < 2^* - 1$ by the classical result of Brezis and Lieb [8]. Actually, Brezis and Lieb [8] consider some systems of equations

$$-\Delta u_i(x) = g^i\big(u(x)\big), \quad i = 1, \ldots, n, \tag{1.2}$$

where the n functions $g^i : \mathbb{R}^n \to \mathbb{R}$ are the gradients of some function $G \in C^1(\mathbb{R}^n)$, namely $g^i(u) = \partial G(u)/\partial u_i$. Under some conditions on g^i (see (2.2), (2.3), (2.4) and (2.8) in [8]), they proved that (1.2) has a ground state solution which belongs to $H^1(\mathbb{R}^N) \cap W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $q < +\infty$ (see Theorem 2.2 and Theorem 2.3 in [8]). Recently, Byeon, Jeanjean and Maris [9] proved that ground state solutions of (1.2) obtained in [8] must be radial up to a translation. Note that (1.1) is a special case of (1.2), and $0 < \lambda < \sqrt{\mu\nu}$ with $1 < p, q < 2^* - 1$ are consistent with those conditions on g^i in [8]; it follows that (1.1) produces a ground state solution in this case. Some related systems of (1.1) in subcritical case were also studied in [12].

Note that in all these works, they only considered the subcritical case $1 < p, q < 2^* - 1$. The critical case for such a system has not ever been studied by variational methods. In this paper, we study the critical case, that is, $1 < p \leqslant q = 2^* - 1$. Since we are concerned with the ground states, we assume $0 < \lambda < \sqrt{\mu\nu}$ in the sequel. This assumption is needed to guarantee that the Nehari manifold is bounded away from $(0, 0)$. Remark that, if $p = q = 2^* - 1$, then system (1.1) has no nontrivial solutions by the Pohozaev identity (see Remark 1.4 below). Thus we only consider the case of $1 < p < q = 2^* - 1$, and system (1.1) turns to be

$$\begin{cases} -\Delta u + \mu u = |u|^{p-1}u + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{2^*-2}v + \lambda u, & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N). \end{cases} \tag{1.3}$$

Let C_{p+1} be the sharp constant of the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \, dx \geqslant C_{p+1} \left(\int_{\mathbb{R}^N} |u|^{p+1} \, dx \right)^{\frac{2}{p+1}}, \tag{1.4}$$

and S the sharp constant of $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geqslant S \left(\int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{\frac{2}{2^*}}. \tag{1.5}$$

Here, $D^{1,2}(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N)\colon\ |\nabla u| \in L^2(\mathbb{R}^N)\}$ with norm

$$\|u\|_{D^{1,2}} := \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{1/2}.$$

Define

$$\mu_0 = \left[\frac{2(p+1)}{N(p-1)} S^{\frac{N}{2}} C_{p+1}^{-\frac{p+1}{p-1}} \right]^{(\frac{p+1}{p-1} - \frac{N}{2})^{-1}}. \tag{1.6}$$

The first main result of the current paper is as follows.

Theorem 1.1. *Assume $N \geqslant 3$, $1 < p < 2^* - 1$ and $\mu, \nu > 0$, $0 < \lambda < \sqrt{\mu\nu}$. Let μ_0 be in* (1.6).

(1) *If $0 < \mu \leqslant \mu_0$, then problem* (1.3) *has a positive ground state (u, v), such that $u, v \in C^2(\mathbb{R}^N, \mathbb{R})$ are both radial symmetric decreasing.*
(2) *If $\mu > \mu_0$, then there exists $\lambda_{\mu,\nu} \in [\sqrt{(\mu - \mu_0)\nu}, \sqrt{\mu\nu})$ such that*
   (i) *if $\lambda < \lambda_{\mu,\nu}$, then problem* (1.3) *has no ground state solutions*;
   (ii) *if $\lambda > \lambda_{\mu,\nu}$, then problem* (1.3) *has a positive ground state (u, v), such that $u, v \in C^2(\mathbb{R}^N, \mathbb{R})$ are both radial symmetric decreasing.*

Remark 1.1. It is interesting that, whether the ground state of (1.3) exists or not depends heavily on the relation of μ, ν, λ and μ_0, and μ_0 can be seen as a critical value of (1.3). In particular, the case $\mu > \mu_0$ is more delicate, and $\lambda_{\mu,\nu}$ can be seen as a critical value in this case. The existence of the ground states for $\lambda = \lambda_{\mu,\nu}$ remains an open question (see Remark 2.1).

Remark 1.2. Though the exact value of C_{p+1} seems unknown, we have the following by-product on the lower bound estimate for C_{p+1}:

$$C_{p+1} > \widetilde{C} := \alpha^{-\alpha} (1 - \alpha)^{-(1-\alpha)} S^{1-\alpha}, \tag{1.7}$$

where

$$\alpha := N \left(\frac{1}{p+1} - \frac{1}{2^*} \right) \in (0, 1).$$

Using (1.6) and (1.7) we can prove that

$$\mu_0 < \bar{\mu}_0 := \alpha(1-\alpha)^{\frac{N-2}{2}(\frac{p+1}{p-1}-\frac{N}{2})^{-1}} < 1, \tag{1.8}$$

see Lemma 2.3.

Remark 1.3. We can also give a estimate from above for $\lambda_{\mu,\nu}$, that is, some number $\tilde{\lambda}_{\mu,\nu} < \sqrt{\mu\nu}$ such that $\lambda_{\mu,\nu} < \tilde{\lambda}_{\mu,\nu}$, see Lemma 2.4 below.

Remark 1.4 *(Nonexistence).* Assume that $\mu, \nu > 0$ and $0 < \lambda < \sqrt{\mu\nu}$. Let (u, v) be a solution of

$$\begin{cases} -\Delta u + \mu u = |u|^{2^*-2}u + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{2^*-2}v + \lambda u, & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N). \end{cases} \tag{1.9}$$

Then one may prove that (u, v) satisfies the Pohozaev identity

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} |\nabla v|^2\,dx = \int_{\mathbb{R}^N} |u|^{2^*}\,dx + \int_{\mathbb{R}^N} |v|^{2^*}\,dx + 2^*\lambda \int_{\mathbb{R}^N} uv\,dx - \frac{2^*}{2}\int_{\mathbb{R}^N} \left(\mu|u|^2 + \nu|v|^2\right) dx.$$

On the other hand,

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \mu|u|^2\right) dx + \int_{\mathbb{R}^N} \left(|\nabla v|^2 + \nu|v|^2\right) dx = \int_{\mathbb{R}^N} |u|^{2^*}\,dx + \int_{\mathbb{R}^N} |v|^{2^*}\,dx + 2\lambda \int_{\mathbb{R}^N} uv\,dx,$$

so

$$\mu\int_{\mathbb{R}^N} |u|^2\,dx + \nu\int_{\mathbb{R}^N} |v|^2\,dx - 2\lambda\int_{\mathbb{R}^N} uv\,dx = 0.$$

This implies $(u, v) \equiv (0, 0)$. Therefore, system (1.9) has no nontrivial solutions.

We can also study some further properties of the ground states obtained in Theorem 1.1. Precisely, we have the following theorem.

Theorem 1.2. *Assume μ, ν, λ satisfy the hypotheses in* (1) *or* (2)(ii) *of Theorem* 1.1. *Let (u, v) be any a ground state of* (1.3) *which exists by Theorem* 1.1, *then up to a translation, $u, v \in$*

$C^2(\mathbb{R}^N,\mathbb{R})$ are positive radial symmetric decreasing. Moreover, there exists a positive constant $C=C(\mu,\nu,\lambda)$ independent of (u,v) such that

$$\|u\|_{L^\infty(\mathbb{R}^N)}+\|v\|_{L^\infty(\mathbb{R}^N)}\leqslant C.$$

Finally, fix any $\mu\in(0,\mu_0)$, $\nu>0$ and let $\lambda_n\in(0,\sqrt{\mu\nu})$ such that $\lambda_n\to 0$ as $n\to+\infty$. Let $(u_{\lambda_n},v_{\lambda_n})$ be any positive radial ground state of (1.3) *with $\lambda=\lambda_n$. Then, passing to a subsequence, $(u_{\lambda_n},v_{\lambda_n})\to(u_0,0)$ strongly in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, where u_0 is a positive radial ground state of*

$$-\Delta u+\mu u=|u|^{p-1}u,\quad u\in H^1\big(\mathbb{R}^N\big).$$

The rest of this paper proves Theorem 1.1 and Theorem 1.2, where we will use the Nehari manifold approach and blowup analysis. For a single equation, blowup analysis has been widely used in the literature, we refer the readers to [11,13,15] and references therein. For systems, blowup analysis was carried out in great generality by Druet and Hebey [14]. Precisely, they studied the following vector valued equations

$$\Delta_g^p\mathcal{U}+A(x)\mathcal{U}=\mathcal{U}^{2^*-1},\tag{1.10}$$

where $\mathcal{U}:M\to\mathbb{R}^p$ is a p-map nonnegative components, Δ_g^p is the Laplace–Beltrami operator acting on p-maps, (M,g) is a smooth compact Riemannian N-manifold. Then the blowup phenomena of (1.10) is well studied in [14]. For related studies, we refer the readers to [14] and references therein.

We give some notations here. Throughout this paper, we denote the norm of $L^p(\mathbb{R}^N)$ by $|u|_p=(\int_{\mathbb{R}^N}|u|^p\,dx)^{\frac{1}{p}}$, and positive constants (possibly different) by C. Denote $X:=H^1(\mathbb{R}^N)$, endowed with the standard scalar product and norm

$$\langle u,v\rangle:=\int\limits_{\mathbb{R}^N}(\nabla u\nabla v+uv)\,dx,\qquad \|u\|^2:=\langle u,u\rangle.$$

Denote $H:=X\times X$ with the norm $\|(u,v)\|^2:=\|u\|^2+\|v\|^2$, and write

$$\|u\|_\mu^2:=\int\limits_{\mathbb{R}^N}\big(|\nabla u|^2+\mu|u|^2\big)\,dx$$

for convenience.

2. Proof of Theorem 1.1 and Theorem 1.2

In this section, we assume that $N\geqslant 3$, $1<p<2^*-1$ and $\mu,\nu>0$, $0<\lambda<\sqrt{\mu\nu}$. Obviously, the bound states of (1.3) are the critical points of the C^2 functional $I_{\mu,\nu,\lambda}:H\to\mathbb{R}$ given by

$$I(u,v):=I_\lambda(u,v):=I_{\mu,\nu,\lambda}(u,v):=\frac{1}{2}\|u\|_\mu^2+\frac{1}{2}\|v\|_\nu^2$$
$$-\frac{1}{p+1}|u|_{p+1}^{p+1}-\frac{1}{2^*}|v|_{2^*}^{2^*}-\lambda\int\limits_{\mathbb{R}^N}uv\,dx.$$

In the sequel, the subscripts μ, ν, λ will be dropped when there is no possible misunderstanding. Define the Nehari manifold corresponding to I as

$$M := M_\lambda := M_{\mu,\nu,\lambda} := \big\{(u,v) \in H \big\backslash \big\{(0,0)\big\}:\ I'(u,v)(u,v) = 0\big\}.$$

Since $\mu, \nu > 0$ and $0 < \lambda < \sqrt{\mu\nu}$, it is easily seen that there exists some $\rho_{\mu,\nu,\lambda} > 0$ such that

$$\|u\|^2 + \|v\|^2 \geqslant \rho_{\mu,\nu,\lambda}, \quad \forall (u,v) \in M_{\mu,\nu,\lambda}.$$

Combining this it is easy to see that M is a complete smooth manifold. Moreover, it is well known that the critical points of I constrained on M are also critical points of I on H and hence solutions of (1.3). Define

$$m_{\mu,\nu,\lambda} := \inf_{(u,v)\in M_{\mu,\nu,\lambda}} I(u,v). \tag{2.1}$$

Then $(u,v) \in M_{\mu,\nu,\lambda}$ such that $I(u,v) = m_{\mu,\nu,\lambda}$ will be a ground state of (1.3).

For any $(u,v) \in H \setminus \{(0,0)\}$, we have

$$\begin{aligned} \max_{t>0} I_\lambda(tu,tv) &= I_\lambda(t_{\lambda,u,v}u, t_{\lambda,u,v}v) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) t_{\lambda,u,v}^{p+1} |u|_{p+1}^{p+1} + \left(\frac{1}{2} - \frac{1}{2^*}\right) t_{\lambda,u,v}^{2^*} |v|_{2^*}^{2^*}, \end{aligned} \tag{2.2}$$

where $t_{\lambda,u,v} > 0$ satisfies $\varphi(\lambda,u,v,t_{\lambda,u,v}) = 0$ and

$$\varphi(\lambda,u,v,t) := \|u\|_\mu^2 + \|v\|_\nu^2 - 2\lambda \int_{\mathbb{R}^N} uv\,dx - t^{p-1}|u|_{p+1}^{p+1} - t^{2^*-2}|v|_{2^*}^{2^*}. \tag{2.3}$$

This implies $(t_{\lambda,u,v}u, t_{\lambda,u,v}v) \in M_\lambda$ and so

$$m_{\mu,\nu,\lambda} = \inf_{(u,v)\in H\setminus\{(0,0)\}} \max_{t>0} I(tu,tv). \tag{2.4}$$

Since $\varphi(\lambda,u,v,t)$ is decreasing with respect to $t > 0$ and $\varphi(\lambda,u,v,0) > 0$, $t_{\lambda,u,v}$ is unique. Moreover, $t_{\lambda,u,v} = 1$ if $(u,v) \in M_\lambda$. Since

$$\max_{t>0} I(tu,tv) \geqslant \max_{t>0} I\big(t|u|, t|v|\big), \tag{2.5}$$

we also have

$$m_{\mu,\nu,\lambda} = \inf_{(u,v)\in H\setminus\{(0,0)\}} \max_{t>0} I\big(t|u|, t|v|\big). \tag{2.6}$$

Lemma 2.1. *For fixed $\mu, \nu > 0$, $m_{\mu,\nu,\lambda}$ is non-increasing with respect to $\lambda > 0$.*

Proof. Let $\lambda_1 < \lambda_2$. Then for any $(u, v) \in H \setminus \{(0,0)\}$ and $t > 0$, we have $I_{\lambda_1}(t|u|, t|v|) \geqslant I_{\lambda_2}(t|u|, t|v|)$. Using (2.6) we get that $m_{\mu,\nu,\lambda_1} \geqslant m_{\mu,\nu,\lambda_2}$. □

Define

$$f_{\beta,\gamma}(u) := \frac{1}{2}\|u\|_\beta^2 - \frac{1}{p+1}\gamma|u|_{p+1}^{p+1}, \qquad g(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \frac{1}{2^*}|v|_{2^*}^{2^*}, \tag{2.7}$$

and denote $f_\beta := f_{\beta,1}$. Then we have the following lemma.

Lemma 2.2. *For any $(u, v) \in H$ with $u \neq 0$ and $v \neq 0$, there holds*

$$\max_{t>0} I(tu, tv) > \min\Big\{\max_{t>0} f_{\mu-\lambda^2/\nu}(tu),\ \max_{t>0} g(tv)\Big\}.$$

Proof. Fix any a pair $(u, v) \in H$ with $u \neq 0$ and $v \neq 0$. Note that $2\lambda uv \leqslant \frac{\lambda^2}{\nu}u^2 + \nu v^2$, we have

$$I(tu, tv) \geqslant f_{\mu-\lambda^2/\nu}(tu) + g(tv).$$

Moreover, there exist $t_1, t_2 > 0$ such that

$$\max_{t>0} f_{\mu-\lambda^2/\nu}(tu) = f_{\mu-\lambda^2/\nu}(t_1 u), \qquad \max_{t>0} g(tv) = g(t_2 v).$$

If $t_1 \geqslant t_2$, then $f_{\mu-\lambda^2/\nu}(t_2 u) > 0$ and $I(t_2 u, t_2 v) > g(t_2 v) = \max_{t>0} g(tv)$. If $t_1 < t_2$, then $g(t_1 v) > 0$ and $I(t_1 u, t_1 v) > f_{\mu-\lambda^2/\nu}(t_1 u) = \max_{t>0} f_{\mu-\lambda^2/\nu}(tu)$. This completes the proof. □

Let w be the radially symmetric positive solution of $-\Delta u + u = u^p$, $u \in H^1(\mathbb{R}^N)$. By [16] we see that w is unique up to a translation and attains the sharp constant C_{p+1} in (1.4), with energy

$$f_1(w) = \left(\frac{1}{2} - \frac{1}{p+1}\right) C_{p+1}^{\frac{p+1}{p-1}},$$

where f_1 is defined in (2.7). Therefore, $w_{\beta,\gamma}(x) := \beta^{\frac{1}{p-1}}\gamma^{-\frac{1}{p-1}} w(\sqrt{\beta}x)$ is the unique positive solution of $-\Delta u + \beta u = \gamma u^p$, $u \in H^1(\mathbb{R}^N)$ with energy

$$f_{\beta,\gamma}(w_{\beta,\gamma}) = \gamma^{-\frac{2}{p-1}}\beta^{\frac{p+1}{p-1}-\frac{N}{2}} f_1(w) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\gamma^{-\frac{2}{p-1}}\beta^{\frac{p+1}{p-1}-\frac{N}{2}} C_{p+1}^{\frac{p+1}{p-1}}. \tag{2.8}$$

Here, $\beta, \gamma > 0$. Denote $w_\beta := w_{\beta,1}$ for convenience. Define $\alpha := N(\frac{1}{p+1} - \frac{1}{2^*}) \in (0, 1)$, then

$$\frac{1}{p+1} = \frac{\alpha}{2} + \frac{1-\alpha}{2^*}.$$

Recall $S, \mu_0, \overline{\mu}_0$ in (1.5), (1.6) and (1.8), we have the following lemma.

Lemma 2.3. *There holds* $0 < \mu_0 < \overline{\mu}_0 < 1$, *and*

$$f_\mu(w_\mu)\begin{cases} > \frac{1}{N}S^{N/2} & \text{if } \mu > \mu_0, \\ = \frac{1}{N}S^{N/2} & \text{if } \mu = \mu_0, \\ < \frac{1}{N}S^{N/2} & \text{if } \mu < \mu_0. \end{cases} \tag{2.9}$$

Proof. By (1.6) and (2.8) we see that $f_{\mu_0}(w_{\mu_0}) = \frac{1}{N}S^{N/2}$. Recall that $p < 2^* - 1$, we have $\frac{p+1}{p-1} - \frac{N}{2} > 0$, and so (2.9) follows directly from (2.8). The fact $\overline{\mu}_0 < 1$ is ensured by $\alpha \in (0,1)$ and the definition (1.8).

It suffices to prove $f_{\overline{\mu}_0}(w_{\overline{\mu}_0}) > \frac{1}{N}S^{N/2}$. For any $u \in H^1(\mathbb{R}^N)$, $u \neq 0$, we see from Hölder inequality and Young inequality that

$$\begin{aligned}\Big(\int_{\mathbb{R}^N} |u|^{p+1}\,dx\Big)^{\frac{2}{p+1}} &\leqslant \Big(\int_{\mathbb{R}^N} |u|^2\,dx\Big)^{\alpha}\Big(\int_{\mathbb{R}^N} |u|^{2^*}\,dx\Big)^{\frac{2}{2^*}(1-\alpha)} \\ &\leqslant \alpha\varepsilon^{1/\alpha}\int_{\mathbb{R}^N} |u|^2\,dx + (1-\alpha)\varepsilon^{-\frac{1}{1-\alpha}}\Big(\int_{\mathbb{R}^N} |u|^{2^*}\,dx\Big)^{\frac{2}{2^*}}.\end{aligned}$$

Choose $C_0 > 0$, $\varepsilon_0 > 0$ such that

$$C_0\alpha\varepsilon_0^{1/\alpha} = 1, \qquad C_0(1-\alpha)\varepsilon_0^{-\frac{1}{1-\alpha}} = S,$$

then we have

$$C_0 = S^{1-\alpha}(1-\alpha)^{-(1-\alpha)}\alpha^{-\alpha}$$

and

$$\|u\|_1^2 > \int_{\mathbb{R}^N} |u|^2\,dx + S\Big(\int_{\mathbb{R}^N} |u|^{2^*}\,dx\Big)^{\frac{2}{2^*}} \geqslant C_0\Big(\int_{\mathbb{R}^N} |u|^{p+1}\,dx\Big)^{\frac{2}{p+1}}.$$

This implies $C_{p+1} > C_0$ by letting $u = w$, that is, (1.7) holds. Combining this with (2.8) we have

$$\begin{aligned}f_\mu(w_\mu) &> \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\mu^{\frac{p+1}{p-1}-\frac{N}{2}}\big(S^{1-\alpha}(1-\alpha)^{-(1-\alpha)}\alpha^{-\alpha}\big)^{\frac{p+1}{p-1}} \\ &= \frac{1}{N}S^{N/2}(1-\alpha)^{\frac{2-N}{2}}\alpha^{\frac{N}{2}-\frac{p+1}{p-1}}\mu^{\frac{p+1}{p-1}-\frac{N}{2}},\end{aligned}$$

which implies $f_{\overline{\mu}_0}(w_{\overline{\mu}_0}) > \frac{1}{N}S^{N/2}$. So $\mu_0 < \overline{\mu}_0$. $\square$

For any $\mu > \mu_0$, $\nu > 0$, we define a C^1 function $h_{\mu,\nu} : (0 + \infty) \to \mathbb{R}$ by

$$h_{\mu,\nu}(a) := \frac{\mu + \nu a^2}{2a} - \frac{\mu_0}{2a}\big(1 + a^2\big)^{-\frac{N}{2}(\frac{p+1}{p-1}-\frac{N}{2})^{-1}}. \tag{2.10}$$

Then

$$h_{\mu,\nu}(a) > \frac{\mu-\mu_0+\nu a^2}{2a} \geqslant \sqrt{(\mu-\mu_0)\nu},$$

and so $h_{\mu,\nu}(a) \to +\infty$ as $a \to 0+$. Meanwhile, $h_{\mu,\nu}$ is increasing with respect to $a \in [\sqrt{\mu/\nu}, +\infty)$. Therefore, there exists $a_{\mu,\nu} \in (0, \sqrt{\mu/\nu})$ such that

$$\tilde{\lambda}_{\mu,\nu} := h_{\mu,\nu}(a_{\mu,\nu}) := \min_{a\in(0,+\infty)} h_{\mu,\nu}(a). \tag{2.11}$$

Note that $h_{\mu,\nu}(\sqrt{\mu/\nu}) < \sqrt{\mu\nu}$, we get that

$$\sqrt{(\mu-\mu_0)\nu} < \tilde{\lambda}_{\mu,\nu} < \sqrt{\mu\nu}. \tag{2.12}$$

Lemma 2.4.

(1) *If $0 < \mu \leqslant \mu_0$, then $m_{\mu,\nu,\lambda} < \frac{1}{N}S^{N/2}$.*
(2) *If $\mu > \mu_0$, then there exists some $\lambda_{\mu,\nu} \in [\sqrt{(\mu-\mu_0)\nu}, \tilde{\lambda}_{\mu,\nu})$, here $\tilde{\lambda}_{\mu,\nu}$ is seen in* (2.11), *such that*
  (i) *if $0 < \lambda \leqslant \lambda_{\mu,\nu}$, then $m_{\mu,\nu,\lambda} = \frac{1}{N}S^{N/2}$;*
  (ii) *if $\lambda_{\mu,\nu} < \lambda < \sqrt{\mu\nu}$, then $m_{\mu,\nu,\lambda} < \frac{1}{N}S^{N/2}$.*

Proof. (1) If $\mu \in (0, \mu_0)$, then it follows from Lemma 2.3 that

$$\max_{t>0} I_{\mu,\nu,\lambda}(tw_\mu, 0) = \max_{t>0} f_\mu(tw_\mu) = f_\mu(w_\mu) < \frac{1}{N}S^{N/2},$$

so $m_{\mu,\nu,\lambda} < \frac{1}{N}S^{N/2}$.

When $\mu = \mu_0$, then $m_{\mu_0,\nu,\lambda} \leqslant f_{\mu_0}(w_{\mu_0}) = \frac{1}{N}S^{N/2}$. Assume that $m_{\mu_0,\nu,\lambda} = \frac{1}{N}S^{N/2}$, then

$$I_{\mu_0,\nu,\lambda}(w_{\mu_0}, 0) = m_{\mu_0,\nu,\lambda}, \quad (w_{\mu_0}, 0) \in M_{\mu_0,\nu,\lambda},$$

which implies that $(w_{\mu_0}, 0)$ is a ground state solution of (1.3). Since $\lambda > 0$, we get that $w_{\mu_0} \equiv 0$, a contradiction. So $m_{\mu_0,\nu,\lambda} < \frac{1}{N}S^{N/2}$.

(2) We fix any $\mu > \mu_0$, $\nu > 0$. First we claim that

$$m_{\mu,\nu,\lambda} = \frac{1}{N}S^{N/2} \quad \text{if } 0 < \lambda \leqslant \sqrt{(\mu-\mu_0)\nu}. \tag{2.13}$$

Assume $0 < \lambda \leqslant \sqrt{(\mu-\mu_0)\nu}$, then $\mu - \frac{\lambda^2}{\nu} \geqslant \mu_0$. Similarly as (2.4), we have

$$f_\mu(w_\mu) = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} f_\mu(tu).$$

By (1.5) we have

$$\inf_{v\in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} g(tv) = \frac{1}{N}S^{N/2}.$$

For any $(u,v)\in H\setminus\{(0,0)\}$, if $v=0$, then $\max_{t>0} I(tu,0)=\max_{t>0} f_\mu(tu)\geqslant \frac{1}{N}S^{N/2}$. If $u=0$, then $\max_{t>0} I(0,tv)\geqslant \max_{t>0} g(tv)\geqslant \frac{1}{N}S^{N/2}$. If $u\neq 0$ and $v\neq 0$, then by Lemma 2.2 and Lemma 2.3 we have

$$\max_{t>0} I(tu,tv)>\min\Big\{\max_{t>0} f_{\mu-\lambda^2/\nu}(tu),\ \max_{t>0} g(tv)\Big\}\geqslant \frac{1}{N}S^{N/2}. \tag{2.14}$$

Combining these with (2.4), we see that $m_{\mu,\nu,\lambda}\geqslant \frac{1}{N}S^{N/2}$. On the other hand, since the equation $-\Delta v+\nu v=|v|^{2^*-2}v$, $v\in H^1(\mathbb{R}^N)$ has no nontrivial solutions by the Pohozaev identity (similarly to Remark 1.4), it is easily seen that S is also the sharp constant (although cannot be attained) of

$$\int_{\mathbb{R}^N} |\nabla v|^2+\nu|v|^2\,dx\geqslant S\bigg(\int_{\mathbb{R}^N} |v|^{2^*}\,dx\bigg)^{\frac{2}{2^*}},$$

which implies that

$$m_{\mu,\nu,\lambda}\leqslant \inf_{v\in H^1(\mathbb{R}^N)\setminus\{0\}}\max_{t>0} I(0,tv)=\frac{1}{N}S^{N/2}, \tag{2.15}$$

and so $m_{\mu,\nu,\lambda}=\frac{1}{N}S^{N/2}$, that is, (2.13) holds.

To prove (i)–(ii), we let $0<\lambda<\sqrt{\mu\nu}$. Recall $a_{\mu,\nu}$ in (2.11), we define

$$\beta:=\frac{\mu+\nu a_{\mu,\nu}^2-2\lambda a_{\mu,\nu}}{1+a_{\mu,\nu}^2},\qquad \gamma:=\frac{1}{1+a_{\mu,\nu}^2}.$$

Then $\beta>0$, $\gamma>0$. It is easy to see from (2.8) that

$$\begin{aligned} m_{\mu,\nu,\lambda}&\leqslant \max_{t>0} I\big(tw_{\beta,\gamma},t(a_{\mu,\nu}w_{\beta,\gamma})\big)\\ &<\big(1+a_{\mu,\nu}^2\big)\max_{t>0} f_{\beta,\gamma}(tw_{\beta,\gamma})=\big(1+a_{\mu,\nu}^2\big)f_{\beta,\gamma}(w_{\beta,\gamma})\\ &=\big(1+a_{\mu,\nu}^2\big)^{\frac{N}{2}}\big(\mu+\nu a_{\mu,\nu}^2-2\lambda a_{\mu,\nu}\big)^{\frac{p+1}{p-1}-\frac{N}{2}}\bigg(\frac{1}{2}-\frac{1}{p+1}\bigg)C_{p+1}^{\frac{p+1}{p-1}}=:A_0. \end{aligned}$$

By Lemma 2.3 we see that $A_0\leqslant \frac{1}{N}S^{N/2}$ is equivalent to

$$\big(1+a_{\mu,\nu}^2\big)^{\frac{N}{2}}\big(\mu+\nu a_{\mu,\nu}^2-2\lambda a_{\mu,\nu}\big)^{\frac{p+1}{p-1}-\frac{N}{2}}\leqslant \mu_0^{\frac{p+1}{p-1}-\frac{N}{2}}.$$

By (2.10) and (2.11), we see that the above inequality is equivalent to $\lambda\geqslant\tilde\lambda_{\mu,\nu}$. Combining this with (2.12), for any $\lambda\in[\tilde\lambda_{\mu,\nu},\sqrt{\mu\nu})$, we have $m_{\mu,\nu,\lambda}<\frac{1}{N}S^{N/2}$. Define

$$\lambda_{\mu,\nu}:=\inf\bigg\{\lambda<\sqrt{\mu\nu}\colon\ m_{\mu,\nu,\tau}<\frac{1}{N}S^{N/2},\ \forall\tau\in[\lambda,\sqrt{\mu\nu})\bigg\}.$$

Then (2.13) implies $\lambda_{\mu,\nu} \in [\sqrt{(\mu-\mu_0)\nu}, \tilde{\lambda}_{\mu,\nu}]$, and for any $\lambda \in (\lambda_{\mu,\nu}, \sqrt{\mu\nu})$, there holds $m_{\mu,\nu,\lambda} < \frac{1}{N}S^{N/2}$, that is, (ii) holds.

We claim that $m_{\mu,\nu,\lambda_{\mu,\nu}} = \frac{1}{N}S^{N/2}$, which implies $\lambda_{\mu,\nu} < \tilde{\lambda}_{\mu,\nu}$ immediately.

By (2.15) we have $m_{\mu,\nu,\lambda_{\mu,\nu}} \leqslant \frac{1}{N}S^{N/2}$. By the definition of $\lambda_{\mu,\nu}$, there exists $\lambda_n < \lambda_{\mu,\nu}$, $n \geqslant 1$ such that

$$\lim_{n\to+\infty} \lambda_n = \lambda_{\mu,\nu}, \qquad m_n := m_{\mu,\nu,\lambda_n} \geqslant \frac{1}{N}S^{N/2}, \quad \forall n \geqslant 1.$$

For any $(u,v) \in H \setminus \{(0,0)\}$, there exists $t_n > 0$ such that $\max_{t>0} I_{\mu,\nu,\lambda_n}(tu,tv) = I_{\mu,\nu,\lambda_n}(t_n u, t_n v)$. Since $\lambda_n \to \lambda_{\mu,\nu}$, we have $t_n \to t_0$ as $n \to +\infty$, where $t_0 > 0$ satisfies $\max_{t>0} I_{\mu,\nu,\lambda_{\mu,\nu}}(tu,tv) = I_{\mu,\nu,\lambda_{\mu,\nu}}(t_0 u, t_0 v)$. Then

$$\limsup_{n\to+\infty} m_n \leqslant \limsup_{n\to+\infty} I_{\mu,\nu,\lambda_n}(t_n u, t_n v) = I_{\mu,\nu,\lambda_{\mu,\nu}}(t_0 u, t_0 v).$$

This implies

$$\frac{1}{N}S^{N/2} \leqslant \limsup_{n\to+\infty} m_n \leqslant m_{\mu,\nu,\lambda_{\mu,\nu}},$$

and so $m_{\mu,\nu,\lambda_{\mu,\nu}} = \frac{1}{N}S^{N/2}$. By Lemma 2.1 and (2.13) we see that (i) holds. This completes the proof. □

The following lemma is concerned with the nonexistence of ground state solutions.

Lemma 2.5. *If $\mu > \mu_0$ and $0 < \lambda < \lambda_{\mu,\nu}$, then problem* (1.3) *has no ground state solutions.*

Proof. Fix any $\nu > 0$ and $\mu > \mu_0$. Assume by contradiction that there exists $\lambda \in (0, \lambda_{\mu,\nu})$ such that (1.3) has a ground state solution $(u_\lambda, v_\lambda) \neq (0,0)$. Then $I_\lambda(u_\lambda, v_\lambda) = m_{\mu,\nu,\lambda} = \frac{1}{N}S^{N/2}$. By (1.3) we see that $u \neq 0$ and $v \neq 0$. By (2.5) and (2.6) we may assume that $u \geqslant 0$, $v \geqslant 0$ (or see the proof of Theorem 1.2 below). By elliptic regularity theory, we see that $u_\lambda, v_\lambda \in C^2(\mathbb{R}^N)$ and so $u_\lambda > 0$, $v_\lambda > 0$ by the strong maximum principle. Take $\lambda_1 \in (\lambda, \lambda_{\mu,\nu})$. Then we see from Lemma 2.4, (2.2) and (2.4) that

$$\begin{aligned}
\frac{1}{N}S^{N/2} &= m_{\mu,\nu,\lambda_1} \leqslant \max_{t>0} I_{\lambda_1}(tu_\lambda, tv_\lambda) \\
&= I_{\lambda_1}(t_{\lambda_1,u_\lambda,v_\lambda} u_\lambda, t_{\lambda_1,u_\lambda,v_\lambda} v_\lambda) \\
&= I_\lambda(t_{\lambda_1,u_\lambda,v_\lambda} u_\lambda, t_{\lambda_1,u_\lambda,v_\lambda} v_\lambda) - (\lambda_1 - \lambda) t^2_{\lambda_1,u_\lambda,v_\lambda} \int\limits_{\mathbb{R}^N} u_\lambda v_\lambda \, dx \\
&< I_\lambda(t_{\lambda_1,u_\lambda,v_\lambda} u_\lambda, t_{\lambda_1,u_\lambda,v_\lambda} v_\lambda) \leqslant I_\lambda(u_\lambda, v_\lambda) \\
&= \frac{1}{N}S^{N/2},
\end{aligned}$$

a contradiction. This completes the proof. □

Lemma 2.6. *Let* $0 < \lambda < \sqrt{\mu\nu}$. *If* $m_{\mu,\nu,\lambda} < \frac{1}{N} S^{N/2}$, *then problem* (1.3) *has a ground state* $(u_0, v_0) \in C^2(\mathbb{R}^N, \mathbb{R})$ *such that* u_0, v_0 *are both positive radial symmetric decreasing with respect to* $r = |x| \in [0, +\infty)$.

Proof. Fix any $\mu, \nu, \lambda > 0$ with $0 < \lambda < \sqrt{\mu\nu}$, and denote $m := m_{\mu,\nu,\lambda} < \frac{1}{N} S^{N/2}$. Let $\varepsilon_n \in (0, 2^* - 1 - p)$ such that $\varepsilon_n \to 0$ as $n \to +\infty$. As has been pointed out in Section 1, by [8] the following subcritical problem

$$\begin{cases} -\Delta u + \mu u = |u|^{p-1} u + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{2^*-2-\varepsilon_n} v + \lambda u, & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{2.16}$$

has a ground state solution $(u_n, v_n) \in H$, with energy $c_n := J_n(u_n, v_n)$. Here

$$J_n(u, v) := \frac{1}{2}\|u\|_\mu^2 + \frac{1}{2}\|v\|_\nu^2 - \frac{1}{p+1}|u|_{p+1}^{p+1} - \frac{1}{2^* - \varepsilon_n}|v|_{2^*-\varepsilon_n}^{2^*-\varepsilon_n} - \lambda \int_{\mathbb{R}^N} uv\, dx.$$

By a similar proof of Theorem 1.3 in [12], we may assume that $u_n > 0$, $v_n > 0$, $u_n, v_n \in C^2(\mathbb{R}^N)$ and u_n, v_n are radial symmetric decreasing. Similarly as (2.4), we have

$$c_n = \inf_{(u,v)\in H\setminus\{(0,0)\}} \max_{t>0} J_n(tu, tv).$$

For any $(u, v) \in H \setminus \{(0,0)\}$, there exists $t_{u,v,n} > 0$ such that $\max_{t>0} J_n(tu, tv) = J_n(t_{u,v,n}u, t_{u,v,n}v)$. Recall $t_{\lambda,u,v}$ in (2.2), it is easily seen that $t_{u,v,n} \to t_{\lambda,u,v}$ as $n \to +\infty$. Thus

$$\limsup_{n\to+\infty} c_n \leqslant \limsup_{n\to+\infty} J_n(t_{u,v,n}u, t_{u,v,n}v) = I(t_{\lambda,u,v}v, t_{\lambda,u,v}v) = \max_{t>0} I(tu, tv).$$

This implies $\limsup_{n\to+\infty} c_n \leqslant m$. So $\{c_n\}_{n\in\mathbb{N}}$ is bounded. Note that

$$\begin{aligned} c_n &= J_n(u_n, v_n) - \frac{1}{p+1} J_n'(u_n, v_n)(u_n, v_n) \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\|u_n\|_\mu^2 + \|v_n\|_\nu^2 - 2\lambda \int_{\mathbb{R}^N} u_n v_n\, dx\right) \\ &\geqslant C\big(\|u_n\|^2 + \|v_n\|^2\big), \end{aligned} \tag{2.17}$$

we get that $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ is bounded in H. Then passing to a subsequence, we may assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in H, and so (u_0, v_0) satisfies (1.3). Since u_n, v_n are radial, we see that u_0, v_0 are radial and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^{p+1}\, dx = \int_{\mathbb{R}^N} |u_0|^{p+1}\, dx. \tag{2.18}$$

Assume that $u_n(0)+v_n(0)=\max_{x\in\mathbb{R}^N}u_n(x)+\max_{x\in\mathbb{R}^N}v_n(x)\to+\infty$ as $n\to+\infty$. We will use a blowup analysis to get a contradiction. Define $K_n:=max\{u_n(0),v_n(0)\}$, then $K_n\to+\infty$. Define

$$U_n(x)=K_n^{-1}u_n\bigl(K_n^{-\alpha_n}x\bigr),\qquad V_n(x)=K_n^{-1}v_n\bigl(K_n^{-\alpha_n}x\bigr),\quad \alpha_n=\frac{2^*-2-\varepsilon_n}{2}.$$

Then $1=\max\{U_n(0),V_n(0)\}=\max\{\max_{x\in\mathbb{R}^N}U_n(x),\max_{x\in\mathbb{R}^N}V_n(x)\}$ and U_n, V_n satisfy

$$\begin{cases}-\Delta U_n+\mu K_n^{-2\alpha_n}U_n=K_n^{p-1-2\alpha_n}U_n^p+\lambda K_n^{-2\alpha_n}V_n, & x\in\mathbb{R}^N,\\ -\Delta V_n+\nu K_n^{-2\alpha_n}V_n=V_n^{2^*-1-\varepsilon_n}+\lambda K_n^{-2\alpha_n}U_n, & x\in\mathbb{R}^N.\end{cases}$$

Since

$$\int_{\mathbb{R}^N}|\nabla U_n|^2\,dx=K_n^{-\frac{(N-2)\varepsilon_n}{2}}\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx,$$

we see that $\{(U_n,V_n)\}_{n\geqslant1}$ is bounded in $D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)=:D$. By elliptic estimates, for a subsequence we have $(U_n,V_n)\to(U,V)\in D$ uniformly in every compact subset of $\mathbb{R}^N$ as $n\to+\infty$, and U, V satisfy

$$-\Delta U=0,\qquad -\Delta V=V^{2^*-1},\quad 0\leqslant U,V\leqslant 1=\max\bigl\{U(0),V(0)\bigr\}.$$

If $U(0)=1$, then by Liouville's theorem we have $U(x)\equiv1$. However,

$$\int_{\mathbb{R}^N}U^{2^*}\,dx\leqslant\lim_{n\to+\infty}\int_{\mathbb{R}^N}U_n^{2^*}\,dx=\lim_{n\to+\infty}K_n^{-N\varepsilon_n/2}\int_{\mathbb{R}^N}u_n^{2^*}\,dx<+\infty,$$

which is a contradiction. So $V(0)=1$, and $V\in D^{1,2}(\mathbb{R}^N)$ is a positive solution of $-\Delta v=|v|^{2^*-2}v$, $v\in D^{1,2}(\mathbb{R}^N)$. This implies $|V|_{2^*}^{2^*}=S^{N/2}$ and so

$$\begin{aligned}\frac1N S^{N/2}&=\frac1N\int_{\mathbb{R}^N}|V|^{2^*}\,dx\leqslant\limsup_{n\to+\infty}\left(\frac12-\frac{1}{2^*-\varepsilon_n}\right)K_n^{\frac{N-2}{2}\varepsilon_n}\int_{\mathbb{R}^N}|V_n|^{2^*-\varepsilon_n}\,dx\\ &=\limsup_{n\to+\infty}\left(\frac12-\frac{1}{2^*-\varepsilon_n}\right)\int_{\mathbb{R}^N}|v_n|^{2^*-\varepsilon_n}\,dx\\ &\leqslant\limsup_{n\to+\infty}\left[\left(\frac12-\frac{1}{2^*-\varepsilon_n}\right)|v_n|_{2^*-\varepsilon_n}^{2^*-\varepsilon_n}+\left(\frac12-\frac{1}{p+1}\right)|u_n|_{p+1}^{p+1}\right]\\ &=\limsup_{n\to+\infty}c_n\leqslant m<\frac1N S^{N/2},\end{aligned}$$

which is also a contradiction. Therefore, $\{u_n,v_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^N)$. This implies that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|v_n|^{2^*-\varepsilon_n}\,dx=\int_{\mathbb{R}^N}|v_0|^{2^*}\,dx. \tag{2.19}$$

In fact, since $(v_n)_{n\geqslant 0}$ are radial and bounded in $H^1(\mathbb{R}^N)$, we see that

$$v_n(x) \to 0 \quad \text{as } |x| \to +\infty \quad \text{uniformly for } n \in \{0\} \cup \mathbb{N}.$$

Therefore, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for any $n \in \{0\} \cup \mathbb{N}$ there holds (let $\varepsilon_0 = 0$)

$$\int\limits_{\mathbb{R}^N \setminus B(0,R_\varepsilon)} |v_n|^{2^*-\varepsilon_n}\,dx \leqslant \varepsilon \int\limits_{\mathbb{R}^N \setminus B(0,R_\varepsilon)} |v_n|^2\,dx \leqslant C\varepsilon,$$

where $B(0,R) := \{x \in \mathbb{R}^N\colon |x| < R\}$. On the other hand, since $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^N)$, the Lebesgue Dominated Convergence Theorem implies

$$\lim_{n\to+\infty} \int\limits_{B(0,R_\varepsilon)} |v_n|^{2^*-\varepsilon_n}\,dx = \int\limits_{B(0,R_\varepsilon)} |v_0|^{2^*}\,dx.$$

So (2.19) holds. Meanwhile, $J_n'(u_n, v_n)(u_n, v_n) = 0$ implies that $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ is bounded away from $(0,0)$ in H. Combining this with (2.17) we see that $\inf_{n\in\mathbb{N}} c_n > 0$. Note that (u_0, v_0) satisfies (1.3), we have

$$\begin{aligned} I(u_0, v_0) &= \left(\frac{1}{2} - \frac{1}{2^*}\right) \int\limits_{\mathbb{R}^N} |v_0|^{2^*}\,dx + \left(\frac{1}{2} - \frac{1}{p+1}\right) \int\limits_{\mathbb{R}^N} |u_0|^{p+1}\,dx \\ &= \lim_{n\to+\infty} \left[\left(\frac{1}{2} - \frac{1}{2^* - \varepsilon_n}\right) |v_n|_{2^*-\varepsilon_n}^{2^*-\varepsilon_n} + \left(\frac{1}{2} - \frac{1}{p+1}\right) |u_n|_{p+1}^{p+1}\right] \\ &= \lim_{n\to+\infty} c_n \in (0, m]. \end{aligned}$$

Therefore, $(u_0, v_0) \neq (0,0)$ and $(u_0, v_0) \in M$, which implies that $I(u_0, v_0) = m = m_{\mu,\nu,\lambda}$, that is, (u_0, v_0) is a ground state of (1.3), and $u_0 \neq 0$, $v_0 \neq 0$. By (2.18) and (2.19) we have

$$\|u_0\|_\mu^2 + \|v_0\|_\nu^2 - 2\lambda \int\limits_{\mathbb{R}^N} u_0 v_0\,dx = \lim_{n\to+\infty} \left(\|u_n\|_\mu^2 + \|v_n\|_\nu^2 - 2\lambda \int\limits_{\mathbb{R}^N} u_n v_n\,dx\right),$$

which implies that $(u_n, v_n) \to (u_0, v_0)$ strongly in H. Since $\{u_n, v_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^N)$, we have $u_0, v_0 \in L^\infty(\mathbb{R}^N)$, and so $u_0, v_0 \in L^q(\mathbb{R}^N)$, $\forall 2 \leqslant q \leqslant +\infty$. Then by elliptic regularity theory, $u_0, v_0 \in C^2(\mathbb{R}^N, \mathbb{R})$. Since u_n, v_n are positive radial symmetric decreasing, we see that $u_0 \geqslant 0$, $v_0 \geqslant 0$ are radial symmetric non-increasing. Then by the strong maximum principle, u_0, v_0 are positive radial symmetric decreasing. □

Proof of Theorem 1.1. Theorem 1.1 follows directly from Lemma 2.4, Lemma 2.5 and Lemma 2.6. □

Remark 2.1. In the case $\mu > \mu_0$, we see from Lemma 2.4 that $m_{\mu,\nu,\lambda_{\mu,\nu}} = \frac{1}{N} S^{N/2}$ and $m_{\mu,\nu,\lambda} < \frac{1}{N} S^{N/2}$ for $\lambda > \lambda_{\mu,\nu}$, and the methods of proving Lemma 2.5 and Lemma 2.6 can't be used in case $\lambda = \lambda_{\mu,\nu}$. Hence, the existence of the ground states for this case remains an open question.

Proof of Theorem 1.2. Assume μ, ν, λ satisfy the hypotheses in (1) or (2)(ii) of Theorem 1.1, and let (u, v) be any a ground state of (1.3). Then $u \neq 0$, $v \neq 0$. Define $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$. Then without loss of generality, we assume that $(u^+, v^+) \neq (0, 0)$. By (2.2) and (2.3) we see that $t_{\lambda,|u|,|v|} \leqslant t_{\lambda,u,v} = 1$, and

$$m := m_{\mu,\nu,\lambda} \leqslant I\big(t_{\lambda,|u|,|v|}|u|, t_{\lambda,|u|,|v|}|v|\big) \leqslant I(u, v) = m.$$

It follows that $t_{\lambda,|u|,|v|} = 1$, that is, $\int_{\mathbb{R}^N} uv\,dx = \int_{\mathbb{R}^N} |u||v|\,dx$. Combining this with $I'(u, v) \times (u^+, v^+) = 0$ we get that $(u^+, v^+) \in M$. Then $I(u, v) \leqslant I(u^+, v^+)$, and so $(u^-, v^-) = (0, 0)$, that is, $u \geqslant 0$, $v \geqslant 0$.

Remark that system (1.3) is cooperative (the definition can be seen in [10]) since $0 < \lambda < \sqrt{\mu\nu}$. Then by [10] we have that (u, v) is radial up to a translation. So we may assume that (u, v) is radial symmetric.

Assume that there exists $(u_n, v_n)_{n\in\mathbb{N}}$ such that they are positive radial symmetric ground states of (1.3) and $u_n(0) + v_n(0) = \max_{x\in\mathbb{R}^N} u_n(x) + \max_{x\in\mathbb{R}^N} v_n(x) \to +\infty$ as $n \to +\infty$. Then by a similar blowup analysis as in Lemma 2.6 we get a contradiction. Thus, there exists a positive constant $C = C(\mu, \nu, \lambda)$ independent of (u, v) such that

$$\|u\|_{L^\infty(\mathbb{R}^N)} + \|v\|_{L^\infty(\mathbb{R}^N)} \leqslant C.$$

By elliptic regularity theory, we have $u, v \in C^2(\mathbb{R}^N, \mathbb{R})$. Hence, by the strong maximum principle, we have $u, v > 0$.

By a similar argument in Lemma 4.5 of [12], we see that u, v are both decreasing with respect to $r = |x| \in [0, +\infty)$.

Finally, fix any $\mu \in (0, \mu_0)$, $\nu > 0$ and let $\lambda_n \in (0, \sqrt{\mu\nu})$ such that $\lambda_n \to 0$ as $n \to +\infty$. Let $(u_{\lambda_n}, v_{\lambda_n})$ be any positive radial ground state of (1.3) with $\lambda = \lambda_n$. By the proof of (1) in Lemma 2.4 we have $m_{\mu,\nu,\lambda_n} \leqslant f_\mu(w_\mu) < \frac{1}{N} S^{N/2}$. Similarly to (2.17), we see that $\{(u_{\lambda_n}, v_{\lambda_n})\}_{n\in\mathbb{N}}$ is bounded in H. Then passing to a subsequence, we may assume that $(u_{\lambda_n}, v_{\lambda_n}) \rightharpoonup (u_0, v_0)$ weakly in H, and so (u_0, v_0) satisfies

$$\begin{cases} -\Delta u + \mu u = |u|^{p-1}u, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{2^*-2}v, & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N). \end{cases}$$

This means $v_0 \equiv 0$. Since u_{λ_n} is radial, one has that u_0 is radial and so

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_{\lambda_n}|^{p+1}\,dx = \int_{\mathbb{R}^N} |u_0|^{p+1}\,dx.$$

We claim that $u_0 \not\equiv 0$. Assume by contradiction that $u_0 \equiv 0$. Similarly as Remark 1.4, by Pohozaev identity and $I'_{\mu,\nu,\lambda_n}(u_{\lambda_n}, v_{\lambda_n})(u_{\lambda_n}, v_{\lambda_n}) = 0$ we have

$$\left(\frac{2^*}{p+1} - 1\right)|u_{\lambda_n}|_{p+1}^{p+1} = \frac{2^* - 2}{2}\left(\mu|u_{\lambda_n}|_2^2 + \nu|v_{\lambda_n}|_2^2 - 2\lambda_n \int_{\mathbb{R}^N} u_{\lambda_n} v_{\lambda_n}\,dx\right).$$

Letting $n \to +\infty$ we have $\mu|u_{\lambda_n}|_2^2 + \nu|v_{\lambda_n}|_2^2 \to 0$. By $I'_{\mu,\nu,\lambda_n}(u_{\lambda_n}, v_{\lambda_n})(0, v_{\lambda_n}) = 0$ we have

$$\lim_{n\to+\infty}\left(\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2\,dx - |v_{\lambda_n}|_{2^*}^{2^*}\right) = 0.$$

By (2.20) below, $\{|v_{\lambda_n}|_{2^*}^{2^*}\}_{n\in\mathbb{N}}$ is bounded. Then, passing to a subsequence, we may assume that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2\,dx = \lim_{n\to+\infty} |v_{\lambda_n}|_{2^*}^{2^*} =: B_0.$$

From Lemma 2.1 we have $\liminf_{n\to+\infty} m_{\mu,\nu,\lambda_n} \geqslant m_{\mu,\nu,\lambda_1} > 0$ and so $B_0 > 0$. Using Sobolev inequality (1.5) we have $B_0 \geqslant \frac{1}{N}S^{N/2}$. Using (2.20) again, we have $\frac{1}{N}B_0 \leqslant f_\mu(w_\mu) < \frac{1}{N}S^{N/2}$, a contradiction. Thus $u_0 \not\equiv 0$ and so

$$\begin{aligned} f_\mu(w_\mu) \leqslant f_\mu(u_0) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)|u_0|_{p+1}^{p+1} \\ &\leqslant \limsup_{n\to+\infty}\left[\left(\frac{1}{2} - \frac{1}{2^*}\right)|v_{\lambda_n}|_{2^*}^{2^*} + \left(\frac{1}{2} - \frac{1}{p+1}\right)|u_{\lambda_n}|_{p+1}^{p+1}\right] \\ &= \limsup_{n\to+\infty} m_{\mu,\nu,\lambda_n} \leqslant f_\mu(w_\mu). \end{aligned} \tag{2.20}$$

This implies $f_\mu(u_0) = f_\mu(w_\mu)$ and so u_0 is a positive radial ground state of

$$-\Delta u + \mu u = |u|^{p-1}u, \quad u \in H^1(\mathbb{R}^N).$$

Moreover, we have $|v_{\lambda_n}|_{2^*}^{2^*} \to 0$ as $n \to +\infty$. Thus

$$\|u_0\|_\mu^2 = \limsup_{n\to+\infty}\left(\|u_{\lambda_n}\|_\mu^2 + \|v_{\lambda_n}\|_\nu^2 - 2\lambda_n \int_{\mathbb{R}^N} u_{\lambda_n} v_{\lambda_n}\,dx\right),$$

which implies $(u_{\lambda_n}, v_{\lambda_n}) \to (u_0, 0)$ strongly in H. This completes the proof. □

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